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Symmetries and Retracts of Quantum Logics

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We prove that there are arbitrarily many quantum logics, none of which is "similar" to a part of another and each of which has the group of all symmetries isomorphic to a given abstract group. Moreover, each of them contains a given logic with atomic blocks as its sublogic.

1. INTRODUCTION AND THE MAIN THEOREM

Every abstract group can be represented as the group of all automorphisms of an orthomodular lattice (see Kalmbach, 1984). We present here results that generalize and strengthen this. A simplified (state-free) version of our Main Theorem can be stated as follows: Given a collection $\{\mathcal{G}_i \mid i \in I\}$ of abstract groups and a partial order $\leq$ on the index set I, then there exists a collection $\{L_i \mid i \in I\}$ of orthomodular lattices such that:

(a) For each $i \in I$, the group of all automorphisms of L_i is isomorphic to $\mathcal{G}_i$.

(b) For each $i, j \in I$, L_i can be embedded into L_j iff $i \leq j$. Moreover, we can require that all the L_i contain a given orthomodular lattice L with atomic blocks. (The choice of a large set I with the discrete order—i.e., any two distinct elements of I are incomparable—gives the "state-free" version of the result.

However, to be closer to the structures investigated in quantum mechanics, we consider quantum logics in the sense of Mackey (1963), i.e. σ-orthomodular posets with a σ-convex full set of states.

First, let us recall the terminology and describe our notation. A *quantum logic* is a pair $Q = (L, M)$, where L is a *σ-orthomodular poset* [i.e., a partial order $\leq$ on L and a complementation $'$: $L \to L$ are given such that L has

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the smallest element 0, the largest element 1, $0 \neq 1$, and $(p')' = p$, $p \vee p' = 1$, $p \wedge p' = 0$ for all $p \in L$, $p \leq q$ iff $p' \geq q'$, $p \leq q$ implies $q = p \vee (q \wedge p')$; moreover, if $p_1, p_2, \ldots$ is a sequence of pairwise orthogonal elements, i.e., $p_i \leq p_j'$ for $i \neq j$, then the join $\bigvee_{n=1}^{\infty} p_n$ exists in L] and M is a *σ-convex full set of states on* L [i.e., each $m \in M$ is a map of L into $\langle 0, 1 \rangle$ such that $m(0) = 0$, $m(p') = 1 - m(p)$, and $m(\bigvee_{n=1}^{\infty} p_n) = \sum_{n=1}^{\infty} m(p_n)$ whenever $p_1, p_2, \ldots$ is a sequence of pairwise orthogonal elements; moreover, M is closed under the forming of σ-convex combinations, i.e., for any sequences $\{\alpha_n\}$ of real numbers and $\{m_n\}$ of states,

$$\alpha_n \geq 0, \qquad \sum_{n=1}^{\infty} \alpha_n = 1 \Rightarrow \sum_{n=1}^{\infty} \alpha_n m_n \in M$$

and M is full in the sense that it determines the order of L, i.e., for every $p, q \in L$, we have, $(\forall m \in M,\ m(p) \leq m(q)) \Rightarrow p \leq q$].

A *sublogic* $Q = (L, M)$ of a quantum logic $\bar{Q} = (\bar{L}, \bar{M})$ is determined by a couple of one-to-one mappings

$$\lambda: \quad L \to \bar{L}, \qquad \mu: \quad M \to \bar{M}$$

where λ is a homomorphism of σ-orthomodular posets (i.e., it preserves 0, complements, and joins of pairwise orthogonal sequences) and μ preserves σ-convex combinations and

(i) $\{\bar{m} \circ \lambda \mid \bar{m} \in \bar{M}\} = M$

(ii) $\mu(m) \circ \lambda = m, \quad \forall m \in M$

The couple (λ, μ) often will be referred to as the *embedding* of Q into $\bar{Q}$.

A quantum logic $Q = (L, M)$ is a *retract* of a quantum logic $\bar{Q} = (\bar{L}, \bar{M})$ if there exist homomorphisms of σ-orthomodular posets

$$c: \quad L \to \bar{L}, \qquad r: \quad \bar{L} \to L$$

such that $r \circ c$ is the identity mapping on L and

$$\forall m \in M, \qquad m \circ r \in \bar{M}$$

$$\forall \bar{m} \in \bar{M}, \qquad \bar{m} \circ c \in M$$

Obviously, if we define $\mu: M \to \bar{M}$ by setting $\mu(m) = m \circ r$, then the couple (c, μ) determines an embedding of Q into $\bar{Q}$ as a sublogic.

A *symmetry* of a quantum logic $Q=(L, M)$ (Pulmannová, 1977) is any automorphism $\tau: L\to L$ for which $\{m\circ\tau \mid m\in M\}=M$.

We recall that a *block* in a σ-orthomodular poset L is every maximal Boolean subalgebra of L (Kalmbach, 1983).

Main Theorem. Let $Q=(L, M)$ be a quantum logic, L having only atomic blocks. Let $\{\mathcal{G}_i \mid i\in L\}$ be any family of groups and let $\preceq$ be a partial order on the index set I. Then there exists a family $\{Q_i \mid i\in I\}$ of quantum logics, $Q_i=(L_i, M_i)$, such that:

(a) For each $i\in I$, the group of all symmetries of Q_i is isomorphic to the given group $\mathcal{G}_i$.

(b) For each $i\in I$, the given quantum logic Q is a sublogic of Q_i.

(c) If $i\preceq j$, then Q_i is a retract of Q_j.

(d) If $i\not\preceq j$, then there is no one-to-one homomorphism of L_i into L_j, so that Q_i is not a sublogic of Q_j.

Remarks

1. It is natural to think of some particular cases, e.g. (a) I is a one-point set [this gives a quantum logic variant of the result of Kalmbach (1984), enriched by the embedding of a given quantum logic]; (b) I is large with the discrete order; (c) I is a long chain.

2. The rest of the paper is devoted to the proof of the Main Theorem. Moreover, we show that the constructed quantum logics $Q_i=(L_i, M_i)$ inherit some nice properties of the given quantum logic $Q=(L, M)$. For example, if L is a lattice, so are L_i, $i\in I$; if Q is *two-valued* (TV) (i.e., every pure state $m\in M$ maps L into the two-point set $\{0, 1\}$), so are Q_i. If Q is *strongly full* (SF) [i.e., for every $a, b\in L$,

$$(\{m\in M \mid m(a)=1\}\subseteq\{m\in M \mid m(b)=1\})\Rightarrow a\leq b]$$

so are Q_i, $i\in I$. We mention explicitly the last two properties in the proofs of the lemmas and propositions in the next parts of the paper.

2. EMBEDDINGS INTO RIGID QUANTUM LOGICS

A quantum logic is called *rigid* if it has no non-identical symmetry. In this section, we construct an embedding of a given quantum logic into a rigid quantum logic.

Every orthomodular poset L is covered by blocks (see Kalmbach, 1983). Following Kalmbach, let us denote by 2^n-block in L any block of L isomorphic to a Boolean algebra with n atoms. A 2^3-block in L is called *clear* if it contains an atom that is dominated by only two non-trivial elements of L [i.e., if x, y, z are its atoms, then one of them, say y, is dominated (besides y and 1) only by x' and z'].

Lemma 1. Let $Q = (L, M)$ be a quantum logic. Then there is a quantum logic $\bar{Q} = (\bar{L}, \bar{M})$ and an embedding (λ, μ) of Q into $\bar{Q}$ such that $\bar{L}$ contains neither a 2^2-block nor a clear 2^3-block. (Moreover, if Q is TV or SF, so is $\bar{Q}$.)

Proof. (a) Every 2^2-block in L, generated by an atom x, is embedded into a 2^4-block, where x becomes one of the atoms; the other atoms, say a, b, c, are newly added to L (we obtain a σ-orthomodular poset $\bar{L}$; $\bar{L}$ is a lattice whenever L is a lattice). We extend each state $m \in M$ to three states $\bar{m}_1, \bar{m}_2, \bar{m}_3$, putting

$$\bar{m}_1(a) = 1 - m(x), \qquad \bar{m}_1(b) = \bar{m}_1(c) = 0$$

$$\bar{m}_2(b) = 1 - m(x), \qquad \bar{m}_2(a) = \bar{m}_2(c) = 0$$

$$\bar{m}_3(c) = 1 - m(x), \qquad \bar{m}_3(a) = \bar{m}_3(b) = 0$$

and $\bar{M}$ is a σ convex hull of the set $\{\bar{m}_1, \bar{m}_2, \bar{m}_3 \mid m \in M\}$. We put, e.g., $\mu(m) = \bar{m}_1$. (Clearly, $\bar{Q}$ is SF or TV if Q is SF or TV.)

(b) Every clear 2^3-block in L with atoms, say, x, y, z, where y is dominated only by x' and z', is embedded into a 2^4-block with atoms x, t, u, z such that $y = t \vee u$, the atoms t, u are newly added to L (we obtain a σ-orthomodular poset $\bar{L}$, which is a lattice whenever L is a lattice). Every state $m \in M$ is extended to two states $\bar{m}_1$ and $\bar{m}_2$ by putting

$$\bar{m}_1(t) = \bar{m}_2(u) = m(y), \qquad \bar{m}_1(u) = \bar{m}_2(t) = 0$$

Then $\bar{M}$ is a σ-convex hull of the set $\{\bar{m}_1, \bar{m}_2 \mid m \in M\}$. We put, e.g., $\mu(m) = \bar{m}_1$. (Clearly, $\bar{Q}$ is SF or TV if Q is SF or TV.)

(c) Repeating the procedures under (a) and (b), we obtain the quantum logic with the required properties.

Remark. In the next proof, we use a construction method of forming orthomodular lattices from undirected graphs [for the idea, see Sabidussi (1957) and Kalmbach (1983)]. An undirected graph $G = (V, E)$ is called *suitable* if it is connected, it contains no triangles and no squares, and each its vertex has the degree at least 2. By $\Phi(G)$ we denote the orthomodular lattice obtained in the following way: every vertex of G is represented by an atom in $\Phi(G)$; every edge $\{x, y\} \in E$ is represented by a clear 2^3-block in $\Phi(G)$ with atoms $x, y, x' \wedge y'$; whenever two edges have a common vertex, the corresponding 2^3-blocks are glued together by the common atom (and its complement). Since G is suitable, $\Phi(G)$ is really an orthomodular lattice (see Kalmbach, 1983). Since every pairwise orthogonal sequence of elements of $\Phi(G)$ contains at most two nonzero elements, $\Phi(G)$ is a σ-orthomodular lattice. Every automorphism of $\Phi(G)$ sends each clear 2^3-block on a clear 2^3-block again. This implies easily that the group Aut $\Phi(G)$ of all automorphisms of $\Phi(G)$ is isomorphic to the group Aut G of all automorphisms of G (Sabidussi 1957; Kalmbach, 1983).

Proposition 1. Every quantum logic $Q=(L, M)$, L having only atomic blocks, can be embedded into a rigid quantum logic $\bar{Q}=(\bar{L}, \bar{M})$.

Proof. By Lemma 1, we can suppose that L contains no 2^2-blocks and no clear 2^3-blocks. Let A be the set of all atoms of L. Let $G=(V, E)$ be a suitable graph such that Aut G is the trivial group and there is an independent set $N \subseteq V$ in G (i.e., $\{x, y\} \notin E$ whenever $x, y \in N$) such that card $N \geq$ card A [such a graph exists; see, e.g., Pultr and Trnková (1980)]. Let $f: A \to N$ be a one-to-one mapping. We form $\bar{L}$ as follows: in the disjoint union $L \dot{\cup} \Phi(G)$ [with 0 in L and 0 in $\Phi(G)$ identified and analogously for 1], we set

$$a \leq f(a)' \qquad \text{for every } a \in A$$

[hence we add $a \vee f(a)$ and $a' \wedge f(a)'$ as new elements].

Every automorphism $\tau: \bar{L} \to \bar{L}$ sends every clear 2^3-block in $\bar{L}$ onto a clear 2^3-block again, every element of $\bar{L}$ that belongs only to clear 2^3-blocks on an element with the same property and every element that belongs also to a block not being a clear 2^3-block on an element with the same property. This implies that τ sends $\Phi(G)$ into itself and L also into itself. Since Aut $\Phi(G) \simeq$ Aut G is trivial, τ must be identical on $\Phi(G)$. Since $a \in A$ is the unique element of $L\backslash\{0\}$ with $a \leq f(a)' = \tau(f(a)')$, necessarily $\tau(a)=a$. Consequently τ is identical on L, hence on the whole $\bar{L}$. Thus, Aut $\bar{L}$ is trivial.

Now, we define the states on $\bar{L}$: for each $m \in M$ and every independent set P of $G=(V, E)$, we define a state m_P on $\bar{L}$ such that

$$m_P(l) = m(l) \qquad \text{for all } l \in L$$

$$m_P(v) = 1 \qquad \text{for all } v \in P\backslash f(A)$$

$$m_P(v) = 1 - m(a) \qquad \text{whenever } v \in P,\ v = f(a) \text{ for some } a \in A,$$

$$m_P(v) = 0 \qquad \text{for all } v \in V\backslash P$$

For the other elements of $\bar{L}$, the value of m_P is determined by the fact that

m_P is a state on $\bar{L}$ [since $m_P(x)+m_P(y)\leq 1$ whenever $\{x, y\}\in E$, the definition of $m_P(x\vee y)$ by $m_P(x)+m_P(y)$ is correct]. The set $\bar{M}$ is just the σ-convex hull of all m_P, where $m\in M$ and P ranging over all independent sets of vertices of G. The routine verification that $\bar{M}$ is a full set of states on $\bar{L}$ is omitted. If $\lambda: L\to \bar{L}$ is the inclusion and $\mu: M\to \bar{M}$ is defined by $\mu(m)=m_{\varnothing}$, then (λ, μ) is an embedding of $Q=(L, M)$ into $\bar{Q}=(\bar{L}, \bar{M})$. And if Q is TV or SF, so is $\bar{Q}$.

Remark. Observe that $\bar{L}\backslash\{0, 1\}$ is a connected poset (in the sense that for every a, b there is a chain $x_0, y_0, \ldots, x_n, y_n$ such that $x_0=a$, $y_n=b$ and $x_i\leq y_i$ for $i=0,\ldots, n$, $y_{i-1}\geq x_i$ for $i=1,\ldots, n$). In fact, suitable graphs are connected; hence, every $x, y\in \Phi(G)\backslash\{0, 1\}$ can be joined by a chain as above and for every element l of $L\backslash\{0, 1\}$ we can find an atom a with $a\leq l$, so that l can be joined with $f(a)'$ in $\Phi(G)$.

Proposition 2. Let $\mathscr{G}$ be an arbitrary group. Let $\bar{Q}=(\bar{L}, \bar{M})$ be a rigid quantum logic, $\bar{L}\backslash\{0, 1\}$ a connected poset. Then there is an embedding of $\bar{Q}$ into a quantum logic $Q^+=(L^+, M^+)$ with the group of all symmetries isomorphic to $\mathscr{G}$. Moreover, if $\bar{Q}$ is TV or SF, so is Q^+.

Proof. Let G be a suitable graph with Aut $G\simeq\mathscr{G}$ such that $\Phi(G)$ is not isomorphic to $\bar{L}$ [since there are arbitrarily large suitable graphs G with Aut $G\simeq\mathscr{G}$ (see Pultr and Trnková, 1980), such a graph exists]. Let L^+ be the disjoint union $L\dot{\cup}\Phi(G)$ [with 0 in $\bar{L}$ and 0 in $\Phi(G)$ identified and analogously for 1]. Then Aut $L^+\simeq\mathscr{G}$. In fact, $\bar{L}\backslash\{0, 1\}$ and $\Phi(G)\backslash\{0, 1\}$ are nonisomorphic connected posets, so every automorphism $\tau\in$ Aut L^+ sends $\bar{L}\backslash\{0, 1\}$ into itself and $\Phi(G)\backslash\{0, 1\}$ also into itself and, since $\bar{Q}$ is rigid, it is identical on $\bar{L}$. The set M^+ of states is obtained by the extensions of elements of $\bar{M}$ as in the previous proof.

3. THE PROOF OF THE MAIN THEOREM

Let $\{\mathscr{G}_i \mid i\in I\}$ be a family of groups and $\leq$ be a partial order on I and we may suppose that for every two elements $i, i'\in I$ there is their meet $i\wedge i'$ in I (it can be easily ensured by enlarging the set I, the corresponding new groups $\mathscr{G}_i$ being defined arbitrarily).

Let us define a small category k as follows: the set obj k of all objects of k is precisely the set I; the set $k(i, i')$ of all morphisms of k from i in i' is

$$k(i, i')=\{[\rho_{i,j}, g, \gamma_{j,i'}] \mid j\in I, j\leq i\wedge i', g\in G_j\}$$

where $\rho_{i,j}$ and $\gamma_{j,i'}$ are symbols making the sets of morphisms disjoint for different pairs of objects. The composition of morphisms in k (which is written for convenience from the left to the right) is defined by

$$[\rho_{i,j}, g, \gamma_{j,i'}] \circ [\rho_{i',j'}, g', \gamma_{j',i''}] = \begin{cases} [\rho_{i,j}, g \cdot g', \gamma_{j,i''}] & \text{if } j = j \wedge j' = j'; \\ [\rho_{i,j}, g, \gamma_{j,i''}] & \text{if } j = j \wedge j' \neq j'; \\ [\rho_{i,j'}, g', \gamma_{j',i''}] & \text{if } j \neq j \wedge j' = j' \\ [\rho_{i,j_o}, 1, \gamma_{j_0,i''}] & \text{if } j_0 = j \wedge j', \quad j_0 \neq j, j_0 \neq j' \end{cases}$$

It is easily seen that this composition is associative, so that we really obtain a category. Let us denote $[\rho_{i,j}, 1, \gamma_{j,j}]$ (*where* 1 *is the unit of the group* $\mathcal{G}_j$) by $r_{i,j}$ and $[\rho_{j,j}, 1, \gamma_{j,i}]$ by $c_{j,i}$. Then, for every $i \in I$, $r_{i,i} = c_{i,i}$ is the identity morphism on the object i, denote it by 1_i. It should not be confusing to denote $[\rho_{i,i}, g, \gamma_{i,i}]$ by g $(\in \mathcal{G}_i)$ again. Hence, we see that the category k is—informally—obtained as follows: for every object $i \in \text{obj } k = I$, we form the endormorphism monoid $k(i, i)$ starting from the group $\mathcal{G}_i$ (we may suppose that the groups are disjoint); if $i \leq j$, we add two "generating" morphisms $c_{i,j} \in k(i, j)$ and $r_{j,i} \in k(j, i)$ and form a "free envelope with respect to the equations"

$$c_{i,j} \circ r_{j,i} = r_{i,i \wedge i'} \circ c_{i \wedge i',i} \qquad \text{for all } i, j, i' \in I, \quad i \leq j \geq i' \tag{1}$$

$$c_{i,j} \circ c_{j,l} = c_{i,l}, \qquad r_{l,j} \circ r_{j,i} = r_{l,i} \qquad \text{for all } i \leq j \leq l \text{ in } I \tag{2}$$

$$c_{i,j} \circ g = c_{i,j}, \qquad g \circ r_{j,i} = r_{j,i} \qquad \text{for all } i < j \text{ in } I \text{ and } g \in \mathcal{G}_j. \tag{3}$$

$$1_i = 1 = c_{i,i} = c_{i,i} \circ 1 = 1 \circ c_{i,i} = 1 \circ r_{i,i} = r_{i,i} \circ 1 = r_{i,i} \tag{4}$$

for all $i \in I$ and $1 \in \mathcal{G}_i$

Observe that every $r_{j,i}$ is a retraction and $c_{i,j}$ the corresponding coretraction (i.e., $c_{i,j} \circ r_{j,i} = 1_i$ for all $i \leq j$ in I) and

if $i \not\leq j$, then every morphism in $k(i, j)$ factors through $r_{i,i \wedge j}$, which is a proper retraction (i.e., not an isomorphism) (*)

Given a cardinal number α, denote by $\mathcal{S}_\alpha$ the category of all suitable graphs (V, E) with card $V \geq \alpha$ and all their compatible mappings as morphisms [i.e., f: $(V, E) \to (V_1, E_1)$ is a morphism of $\mathcal{S}_\alpha$ iff it is a mapping of V into V_1 such that $\{x, y\} \in E \Rightarrow \{f(x), f(y)\} \in E_1$]. By Pultr and Trnková (1980, Chapter IV), every small category can be fully embedded in $\mathcal{S}_\alpha$. Denote by $\Psi : k \to \mathcal{S}_\alpha$ a full embedding [i.e., for all objects i, j of k, Ψ maps bijectively $k(i, j)$ onto the set of all compatible maps of $\Psi(i)$ into $\Psi(j)$]. If $i, j \in I$, $i \leq j$, then $r_{j,i} \in k(j, i)$ is a retraction in k; hence $\Psi(r_{j,i})$ is a retraction in $\mathcal{S}_\alpha$, so it is surjective on vertices as well as on edges. But for $i < j$, $r_{j,i}$ is not an isomorphism and therefore $\Psi(r_{j,i})$ is not one-to-one. Consequently, by (*), if $i \not\leq j$, then there is no one-to-one compatible mapping of the graph $\Psi(i)$ into $\Psi(j)$. Moreover, Aut $\Psi(i) \simeq \mathcal{G}_i$ for every $i \in I$.

The completion of the proof of the Main Theorem is now at hand. Given a quantum logic $Q = (L, M)$, L having only atomic blocks, construct a rigid quantum logic $\bar{Q} = (\bar{L}, \bar{M})$ as in the proof of Proposition 1. Let k be the small category constructed from $\{\mathcal{G}_i \mid i \in I\}$ and the partial order $\leq$ on I. Choose $\alpha >$ card $\bar{L}$ and find a full embedding Ψ: $k \to \mathcal{S}_\alpha$. Then, for each $i \in I$, construct the quantum logic $Q_i = (L_i, M_i)$ with Aut $L_i \simeq \mathcal{G}_i$ as in the proof of Proposition 2 by means of $\bar{Q}$ and the suitable graph $\Psi(i)$. If $i \leq j$, then the graph $\Psi(i)$ is a retract of $\Psi(j)$ and this implies easily that Q_i is a retract of Q_j. Of $1 \not\leq j$, then there is no one-to-one homomorphism of L_i into L_j. In fact, L_i is obtained from $\bar{L} \dot{\cup} \Phi(\Psi(i))$ and L_j from $\bar{L} \dot{\cup} \Phi(\Psi(j))$ (see the proof of Proposition 2), so any one-to-one homomorphism $L_i \to L_j$ sends the connected poset $\Phi(\Psi(i)) \backslash \{0, 1\}$ either into $\bar{L} \backslash \{0, 1\}$ or into $\Phi(\Psi(j)) \backslash \{0, 1\}$. The first case is impossible because $\alpha >$ card $\bar{L}$, the second case is also impossible because if $i \not\leq j$, then there is no one-to-one compatible mapping of $\Psi(i)$ into $\Psi(j)$ and hence no one-to-one homomorphism of $\Phi(\Psi(i))$ into $\Phi(\Psi(j))$ [in fact, if h: $\Phi(\Psi(i)) \to \Phi(\Psi(j))$ is a one-to-one homomorphism, then it sends every chain $0 < x < x \vee y < 1$ of the length 4 on a chain of the length 4 in $\Phi(\Psi(j))$, say $0 < a < b < 1$, so that $a = h(x)$ is either a vertex of $\Psi(j)$ or an atom of the form $c' \wedge d'$; but the last case is impossible because there are at most two such chains containing $c' \wedge d'$, namely $0 < c' \wedge d' < c' < 1$ and $0 < c' \wedge d' < d' < 1$, while there are at least four such chains containing x, namely $0 < x < x \vee y < 1$, $0 < x < y' < 1$, $0 < x < x \vee z < 1$, and $0 < x < z' < 1$, where $\{x, y\}$ and $\{x, z\}$ are distinct edges with the vertex x in the suitable graph $\Psi(i)$; consequently, h sends vertices of $\Psi(i)$ on vertices of $\Psi(j)$ and if $\{x, y\}$ is an edge of $\Psi(i)$, then $h(x \vee y) = h(x) \vee h(y)$, so that $\{h(x), h(y)\}$ is an edge of $\Psi(j)$].

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